A simple proof of Wehrl's conjecture on entropy

2000 J. Phys. A: Math. Gen. 33 3093

(http://iopscience.iop.org/0305-4470/33/16/303)

J. Phys. A: Math. Gen. **33** (2000) 3093–3096. Printed in the UK PII: S0305-4470(00)12221-8

A simple proof of Wehrl's conjecture on entropy

Shunlong Luo
Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, Academia Sinica, Beijing, 100080 People's Republic of China
E-mail: luosl@amath4.amt.ac.cn

Received 1 March 2000

Abstract. Based on the hypercontractivity of the free harmonic oscillator Hamiltonian and the canonical commutation relation, we present a simple proof of Wehrl's conjecture on the Shannon entropy of phase space probability densities in the Fock–Bargmann representation.

1. Introduction

Shannon entropy is a measure of the uncertainty inherent in a preassigned probability scheme. For a probability density $\rho(m)$ on some measure space $(M, \mathrm{d}m)$, its Shannon entropy is defined as

$$S(\rho) = -\int_M \rho(m) \ln \rho(m)\, \mathrm{d}m.$$

In the Schrödinger representation of a quantum harmonic oscillator with one degree of freedom, the quantum state space is $L^2(R, \mathrm{d}x)$. Any $f \in L^2(R, \mathrm{d}x)$ with unit norm is called a wavefunction and $|f(x)|^2$ is interpreted as the probability density of the position observable (or the momentum observable). Its Shannon entropy is

$$S(f) = -\int_R |f(x)|^2 \ln |f(x)|^2\, \mathrm{d}x.$$

In particular, when f is a Gaussian wavepacket

$$f(x) = \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right]\right)^{1/2}$$

its Shannon entropy is $S(f) = \frac{1}{2} + \ln(\sqrt{2\pi}\sigma)$. Consequently, $S(f)$ has a range $(-\infty, \infty)$ when f varies in $L^2(R, \mathrm{d}x)$, and $S(f)$ does not have a lower bound.

In the Fock–Bargmann representation (Bargmann 1961), the quantum harmonic oscillator state space is

$$H^2(C) = \left\{ f : C \to C, \text{holomorphic}, \langle f, f\rangle := \int_C f(z)\overline{f(z)}\, \mathrm{d}\mu(z) < \infty \right\}.$$

Here $\mathrm{d}\mu(z) = \pi^{-1} \mathrm{e}^{-z\bar{z}}\, \mathrm{d}z\, \mathrm{d}\bar{z}$ is the standard Gaussian measure on C. The Bargmann transform

$$Bf(z) = \int_R b(z, x) f(x)\, \mathrm{d}x$$

establishes an isometry from $L^2(R, \mathrm{d}x)$ onto $H^2(C)$ and intertwines the Schrödinger representation and the Fock–Bargmann representation. Here

$$b(z, x) = (2\pi)^{-1/4} \exp\{-z^2/2 + zx - x^2/4\}.$$

Any $f \in H^2(C)$ with unit norm is interpreted as a (Bargmann) wavefunction, and $\rho_f(z) = |f(z)|^2 \mathrm{e}^{-|z|^2}$ is interpreted as a phase space probability density on the phase space $(C, \pi^{-1}\,\mathrm{d}z\,\mathrm{d}\bar{z})$. Wehrl (1979) proposed to use the Shannon entropy of this probability as a classical entropy, and conjectured that

$$S_W(f) := -\int_C \rho_f(z) \ln \rho_f(z) \pi^{-1}\,\mathrm{d}z\,\mathrm{d}\bar{z} \geqslant 1$$

for any $f \in H^2(C)$, $\|f\|_2 = 1$. The equality is achieved for normalized coherent states, that is, when f is of the form $f(z) = \mathrm{e}^{-|\xi|^2/2+\bar{\xi}z}$ for some $\xi \in C$. The above inequality is in the spirit of Heisenberg's uncertainty principle.

Wehrl's conjecture was first proved by Lieb (1978), who used two deep results in harmonic analysis: the strengthened Hausdorff–Young inequality and the sharp Young inequality. In view of the Heisenberg group lying behind the coherent states, Lieb (1994) invited a simpler group-theoretic proof of Wehrl's conjecture. In this paper, we present such a proof based directly on hypercontractivity and the canonical commutation relation.

2. Hypercontractivity and Wehrl's conjecture

First, we review some fundamental facts about the Bargmann space and operators on it. $H^2(C)$ has a reproducing kernel $e_\xi(z) := \mathrm{e}^{\bar{\xi}z}$, that is, for any $f \in H^2(C)$, it holds that

$$f(z) = \int_C \mathrm{e}^{\bar{\xi}z} f(\xi)\,\mathrm{d}\mu(\xi) \qquad \text{for any } z \in C.$$

The Schwarz inequality implies that

$$|f(z)| \leqslant \mathrm{e}^{|z|^2/2} \|f\|_2.$$

Thus if $\|f\|_2 = 1$, then

$$\rho_f(z) = |f(z)| \mathrm{e}^{-|z|^2/2} \leqslant 1.$$

This is a kind of uncertainty relation, it manifests the limit of concentration of phase space measurement. Wehrl's conjecture on entropy sets a more synthesized uncertainty relation.

The creation operator a^- and annihilation operator a^+ are defined as

$$a^- f(z) = \frac{\partial}{\partial z} f(z) \qquad a^+ f(z) = z f(z)$$

respectively. They are adjoint to each other. Clearly, $a^- e_\xi(z) = \bar{\xi} e_\xi(z)$, and the (Heisenberg) canonical commutation relation holds:

$$[a^-, a^+] := a^- a^+ - a^+ a^- = I.$$

The free Hamiltonian after subtraction of the zero-point energy is

$$N = a^+ a^- = z \frac{\partial}{\partial z}.$$

The L_p norm is defined as

$$\|f\|_p = \left(\int_C |f(z)|^p \, \mathrm{d}\mu(z)\right)^{1/p}.$$

The following hypercontractivity property for a harmonic oscillator free Hamiltonian, which is a strengthened version of Nelson's famous hypercontractivity result, was proved by Janson (1983) and Zhou (1991).

Lemma. *Let* $T_t = \mathrm{e}^{-tN}$, $t \in [0, \infty)$, *be the dynamical semigroup generated by* N, $2 \leqslant p \leqslant q$, *then* $\|T_t f\|_q \leqslant \|f\|_p$ *if and only if* $\mathrm{e}^{-2t} \leqslant p/q$. *The equality holds only when* $f = \alpha \mathrm{e}^{\bar{\xi} z}$ *for some* $\alpha, \xi \in C$.

Theorem (Wehrl, Lieb). *For any* $f \in H^2(C)$, $\|f\|_2 = 1$, *it holds that*

$$S_W(f) \geqslant 1.$$

Moreover, the equality is achieved only for coherent states, that is, when f *is of the form* $f(z) = \mathrm{e}^{-|\xi|^2/2 + \bar{\xi} z}$ *for some* $\xi \in C$.

Proof. The result follows directly from the hypercontractivity and the canonical commutation relation. In fact, from the lemma, set $p = 2$, $q = 2\mathrm{e}^{2t}$, and taking logarithms, we have

$$\ln \|T_t f\|_{2\mathrm{e}^{2t}} - \ln \|f\|_2 \leqslant 0.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \ln \|T_t f\|_{2\mathrm{e}^{2t}} = \lim_{t \to 0} \frac{\ln \|T_t f\|_{2\mathrm{e}^{2t}} - \ln \|f\|_2}{t} \leqslant 0.$$

However,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \ln \|T_t f\|_{2\mathrm{e}^{2t}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\frac{1}{2\mathrm{e}^{2t}} \ln \int_C |f(\mathrm{e}^{-t} z)|^{2\mathrm{e}^{2t}} \, \mathrm{d}\mu(z)\right) \\
&= -\ln \|f\|_2^2 + \frac{1}{2\|f\|_2^2} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_C |f(\mathrm{e}^{-t} z)|^{2\mathrm{e}^{2t}} \, \mathrm{d}\mu(z) \\
&= \tfrac{1}{2} \int_C \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{\mathrm{e}^{2t} \ln(f(\mathrm{e}^{-t} z)\overline{f(\mathrm{e}^{-t} z)})} \, \mathrm{d}\mu(z) \\
&= \tfrac{1}{2} \int_C |f(z)|^2 \left(2 \ln |f(z)|^2 + |f(z)|^{-2} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f(\mathrm{e}^{-tz})\overline{f(\mathrm{e}^{-t} z)})\right) \mathrm{d}\mu(z) \\
&= \int_C |f(z)|^2 \ln |f(z)|^2 \, \mathrm{d}\mu(z) \\
&\quad -\tfrac{1}{2} \int_C \left((a^+ a^- f)(z)\overline{f(z)} + \overline{(a^+ a^- f)(z)} f(z)\right) \mathrm{d}\mu(z) \\
&= \int_C |f(z)|^2 \ln |f(z)|^2 \, \mathrm{d}\mu(z) - \langle a^+ a^- f, f\rangle.
\end{aligned}$$

Consequently,

$$-\int_C |f(z)|^2 \ln |f(z)|^2 \, \mathrm{d}\mu(z) \geqslant -\langle a^+ a^- f, f\rangle$$

and

$$
\begin{aligned}
S_W(f) &= -\int_C |f(z)|^2 e^{-|z|^2} \ln\left(|f(z)|^2 e^{-|z|^2}\right) \pi^{-1}\, dz\, d\bar{z} \\
&= -\int_C |f(z)|^2 \ln |f(z)|^2\, d\mu(z) + \int_C |z|^2 |f(z)|^2\, d\mu(z) \\
&\geqslant \int_C |z|^2 |f(z)|^2\, d\mu(z) - \langle a^+ a^- f, f\rangle \\
&= \langle a^+ f, a^+ f\rangle - \langle a^+ a^- f, f\rangle \\
&= \langle (a^- a^+ - a^+ a^-) f, f\rangle \\
&= \langle f, f\rangle = 1.
\end{aligned}
$$

The equality holds when f is a coherent state, as implied by the case of hypercontractivity. □

3. Discussion

In the Schrödinger representation, a wavefunction only describes the probability of the position observable or the momentum observable, but not both, and the corresponding Shannon entropy does not have a lower bound. However, in the Fock–Bargmann representation, a wavefunction describes the phase space probability, and thus in a certain sense is a 'joint probability' of the position observable and the momentum observable. The Wehrl–Lieb theorem on the lower bound of the Shannon entropy is physically in the spirit of Heisenberg's uncertainty principle. Actually, in the Schrödinger representation, there is also a result in this spirit:

$$S(f) + S(\widehat{f}) \geqslant 1 + \ln \pi$$

for any $f \in L^2(R, dx)$ with a unit norm. Here

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_R e^{-itx} f(x)\, dx$$

is the Fourier transform of f (see Ohya and Petz (1993) for a proof).

Finally, we mention that a similar statement, Lieb's conjecture for the Wehrl entropy of Bloch coherent states, is still open. See Lieb (1994) and Schupp (1999). The latter obtained a partial result.

References

Bargmann V 1961 On a Hilbert space of holomorphic functions and an associated integral transform *Commun. Pure Appl. Math.* **14** 187–214

Janson S 1983 On hypercontractivity for multipliers on orthogonal polynomials *Ark. Math.* **21** 91–110

Lieb E H 1978 Proof of an entropy conjecture of Wehrl *Commun. Math. Phys.* **62** 35–41

——1994 Coherent states as a tool for obtaining rigorous bounds *Coherent States: Past, Present and Future* ed D H Feng and J Klauder (Singapore: World Scientific) pp 267–78

Ohya M and Petz D 1993 *Quantum Entropy and its Use* (Berlin: Springer)

Schupp P 1999 On Lieb's conjecture for the Wehrl entropy of Bloch coherent states *Commun. Math. Phys.* **207** 481–93

Wehrl A 1979 On the relation between classical and quantum mechanical entropy *Rep. Math. Phys.* **16** 353–8

Zhou Z F 1991 The contractivity of the free Hamiltonian semigroup in the L_p spaces of entire functions *J. Func. Anal.* **96** 407–25